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Expectation values and uncertainties of radial and angular variables for a three-dimensional coherent oscillator

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Abstract. We calculate expectation values and uncertainties of the radial coordinate and momentum, and the azimuthal angle and angular momentum components for a three-dimensional harmonic oscillator in a coherent state. The operators corresponding to powers of the angular variable are defined by their Fourier expansions. For large distances of the particle from the origin, where the oscillator becomes classical, radial and angular uncertainty products approach the value $\hbar/\sqrt{2}$, except for the point of discontinuity, where the angular uncertainty becomes infinite. For small distances, where the oscillator tends to its quantum ground state, the radial uncertainty stays finite and the angular uncertainty product tends to zero. It is also shown that the concept of a polar momentum operator is meaningless.

1. Introduction

1.1. General remarks

The problem of phase or angular uncertainties in quantum mechanics is almost as old as the Heisenberg uncertainty principle itself (cf Jordan 1927). It culminates in the fact that the continuous variable ϕ does not correspond to a hermitian operator in the space of periodic wavefunctions. Hence the uncertainty relation derived from Schwartz's inequality (Weyl 1928, Messiah 1964)

$$\Delta A \, \Delta B \geqslant \tfrac{1}{2}|\langle[A, B]\rangle|, \tag{1.1}$$

where A and B are hermitian operators and the variances are defined by

$$\Delta A = (\langle A^2\rangle - \langle A\rangle^2)^{1/2}, \tag{1.2}$$

does not hold for the angular variable ϕ and the operator of the z component of the angular momentum,

$$l_z = (\hbar/\mathrm{i})(\partial/\partial\phi). \tag{1.3}$$

This difficulty can be circumvented in two ways. Either the Hilbert space is enlarged with the help of spinor-like wavefunctions (Moshinsky and Seligman 1978, Newton 1980), or the continuous angle ϕ is replaced by a periodic function of ϕ. While Louisell (1963) and Lévy-Leblond (1976) favour the trigonometric functions $\cos\phi$, $\sin\phi$, $e^{\pm i\phi}$, Nieto (1967), Judge (1963, 1964), Judge and Lewis (1963) and Susskind and Glasgower (1964) employ the sawtooth-like Fourier expansion of ϕ. For a review on the problem of phase and angular variables see Carruthers and Nieto (1968), and for recent educational notes see Harris and Strauss (1978) and Roy and Sannigrahi (1979). Whereas the former functions are appealing from the mathematical point of view, the experimental situation conforms to the sawtooth definition. Without *a priori* knowledge of the history of an angular motion, an angle can only be measured with the aid of a protractor within a range $\Phi_0 \ldots \Phi_0+2\pi$. Here Φ_0 is an arbitrary angle chosen in such a way as to minimise the influence of the discontinuity on the result. In a small-angle scattering experiment, for instance, zero degrees is defined by the forward beam direction with positive angles resulting, for example, in Rutherford scattering, and negative angles, in certain cases, obtained by an attractive potential. Here $\Phi_0=-\pi$ evidently is favoured by the experiment. In uniform angular motion, on the other hand, as for instance realised by a three-dimensional harmonic oscillator, the influence of the discontinuity cannot be eliminated whatever value for Φ_0 is taken. It becomes overwhelming if the wavefunction is spread mainly about the origin.

In this paper we follow Nieto (1967) and define the hermitian operators corresponding to ϕ and ϕ^2 by their Fourier expansions, and calculate various expectation values, variances and uncertainty products of operators associated with angles and angular momenta for a three-dimensional coherent oscillator. In doing this we complete the studies of Carruthers and Nieto (1968) who first observed that $\Delta\phi\,\Delta l_z$ stays finite as the distance of the particle from the origin becomes large: here we show that this uncertainty product approaches $\hbar/\sqrt{2}$. The relevant quantities will be given for all distances, and the limits of small and large distances are studied.

Furthermore, since the radial coordinate r is restricted to positive values, the radial momentum operator

$$p_r=\frac{\hbar}{\mathrm{i}}r^{-1}\frac{\partial}{\partial r}r \tag{1.4}$$

favoured by Messiah (1964) is not hermitian. As shown by Liboff *et al* (1973), however, it is an admissible observable equivalent provided the condition for the wavefunction

$$\sqrt{r}\psi(r)\to 0 \qquad \text{for } r\to 0 \tag{1.5}$$

holds. Since this is the case in our example, we also calculate expectation values, variances and uncertainty products associated with the radial variable and momentum. As a result we derive that $\Delta r\,\Delta p_r$ also approaches $\hbar/\sqrt{2}$ for large distances.

As concerns the polar angle θ, Blochinzew (1972) suggested a polar momentum operator p_θ. It will be shown that the expectation value of p_θ^2 is infinite for our example. This definition of p_θ is therefore not useful.

1.2. The coherent oscillator

The time-dependent Schrödinger equation for a three-dimensional isotropic harmonic oscillator reads

$$\mathrm{i}\hbar\dot{\psi}=[-(\hbar^2/2m)\nabla^2+\tfrac{1}{2}m\omega^2r^2]\psi. \tag{1.6}$$

The classical oscillator revolving with constant angular velocity ω and angular momentum L,

$$\Phi(t)=\omega t, \tag{1.7a}$$

$$L=m\omega R^2, \tag{1.7b}$$

on a circle of constant radius R in the plane of polar angle $\Theta=\pi/2$, is represented quantum mechanically by the coherent Gaussian wave packet solution of equation (1.6),

$$\psi(r,t)=(2\pi\chi)^{-3/4}\exp[-(1/4\chi)(r^2+R^2-2rR\sin\theta\, e^{i(\phi-\Phi)})-\tfrac{1}{2}i\Phi]. \tag{1.8}$$

Here the spherical coordinates are restricted to $0\leqslant r<\infty$, $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi\leqslant2\pi$, and χ is the square of the minimum position uncertainty in every direction,

$$(\Delta x)^2=(\Delta y)^2=(\Delta z)^2=\chi=\hbar/2m\omega. \tag{1.9}$$

This wave packet will be employed in the following for the computation of the relevant expectation values. Introducing the dimensionless radius

$$\rho=R/(2\chi)^{1/2}, \tag{1.10}$$

the wave packet becomes classical for $\rho\to\infty$, and the quantum ground state limit is given by $\rho\to0$.

2. Radial uncertainties

2.1. Radial variable

In dealing with expectation values of functions $A(r)$ of the radial coordinate only, integration over ϕ can be performed to yield a modified Bessel function of integer order (AS9.6.16)† which, in turn, can be integrated over θ to yield a product of modified spherical Bessel functions (GR6.681.8)†. Hence

$$\langle A(r)\rangle=\frac{1}{R(2\pi\chi)^{1/2}}\int_0^\infty dr\, rA(r)\left[\exp\left(-\frac{(r-R)^2}{2\chi}\right)-\exp\left(-\frac{(r+R)^2}{2\chi}\right)\right]. \tag{2.1}$$

This integral exists for powers $A(r)=r^n$, $n\geqslant-2$ (GR3.462.1), with the result

$$\langle r^n\rangle=\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{n+3}{2}\right)(2\chi)^{n/2}M\left(-\frac{n}{2},\frac{3}{2};-\rho^2\right),\qquad n\geqslant-2. \tag{2.2}$$

Here $M(a,b;x)$ is the regular confluent hypergeometric function (AS13.1.2). Special cases of (2.2) are:

$$\langle r^{-2}\rangle=D(\rho)/\chi \tag{2.3}$$

where

$$D(\rho)=\frac{1}{\rho}\exp(-\rho^2)\int_0^\rho dx\,\exp(x^2) \tag{2.4}$$

is Dawson's integral;

$$\langle r^{-1}\rangle=\mathrm{erf}(\rho)/R, \tag{2.5}$$

$\mathrm{erf}(x)$ being the error function (AS7.1.1);

$$\langle r\rangle=R(1+1/2\rho^2)\,\mathrm{erf}(\rho)+(2\chi/\pi)^{1/2}\exp(-\rho^2); \tag{2.6}$$

† Formulae from Abramowitz and Stegun (1965) are referred to by numbers with prefix AS, and those from Gradshteyn and Ryzhik (1965) with prefix GR.

and

$$\langle r^2 \rangle = R^2 + 3\chi. \tag{2.7}$$

Furthermore, equation (2.2) simplifies to a polynomial for all even non-negative powers (AS13.6.18):

$$\langle r^{2n} \rangle = [(-\chi/2)^n / 2\mathrm{i}\rho] H_{2n+1}(\mathrm{i}\rho), \tag{2.8}$$

where $H_n(x)$ is the Hermite polynomial.

The expectation value of the potential energy can now be deduced from equation (2.7):

$$\langle V(r) \rangle = \tfrac{1}{2} m\omega^2 \langle r^2 \rangle = \tfrac{1}{2} m\omega^2 R^2 + \tfrac{3}{4}\hbar\omega. \tag{2.9}$$

In addition to the classical potential energy, according to the virial theorem, it also contains half of the zero-point energy of the oscillator.

2.2 Radial momentum

Expectation values of the radial momentum operator and powers thereof are obtained by evaluating integrals similar to (2.1). First we note that

$$\langle p_r \rangle = 0 \tag{2.10}$$

because R is a constant of motion. The only quantity of further relevance is

$$\langle p_r^2 \rangle = (\hbar^2/2\chi)(1 + \tfrac{1}{2}D(\rho)), \tag{2.11}$$

from which we obtain the expectation value of the radial kinetic energy,

$$\langle T_{\mathrm{rad}} \rangle = (1/2m)\langle p_r^2 \rangle = \tfrac{1}{4}\hbar\omega(1 + \tfrac{1}{2}D(\rho)). \tag{2.12}$$

The uncertainties of radial position and momentum can now be calculated with the help of equations (2.6), (2.7), (2.10) and (2.11). Since all quantities are smooth functions of the distance R, the uncertainty product as shown in figure 1 is also smooth. Limiting values for small and large distances will be derived in the next subsections.

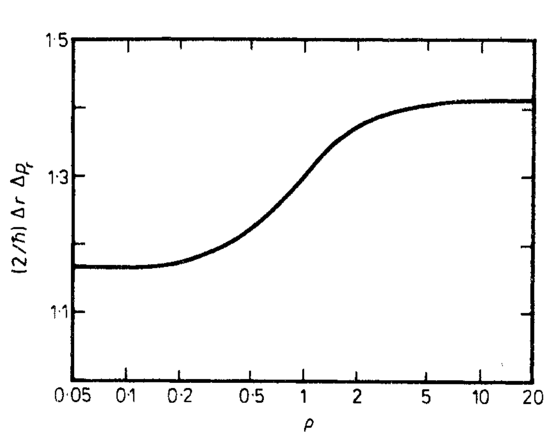

Figure 1. Uncertainty product in units of $\hbar/2$ of the radial variable and momentum, plotted against the classical dimensionless radius $\rho = R/(2\chi)^{1/2}$.

2.3. Small radii

By definition of the non-negative variable r, its expectation value does not vanish as $R \to 0$. From the series expansion of the confluent hypergeometric function (AS13.1.2),

$$\langle r^n \rangle = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{n+3}{2}\right) (2\chi)^{n/2} \left(1 + \frac{n}{3}\rho^2 + \frac{(n-2)n}{30}\rho^4 + \ldots\right), \tag{2.13}$$

one gets

$$\langle r \rangle = 2(2\chi/\pi)^{1/2}(1 + \tfrac{1}{3}\rho^2 + \ldots) \tag{2.14}$$

and, together with the exact expression (2.7),

$$(\Delta r)^2 = \chi(3 - 8/\pi)(1 + \tfrac{2}{3}\rho^2 + \ldots).$$

Dawson's integral has a similar series expansion resulting in

$$\langle p_r^2 \rangle = (\Delta p_r)^2 = (3\hbar^2/4\chi)(1 - \tfrac{2}{9}\rho^2 + \ldots), \tag{2.15}$$

$$\langle T_{\mathrm{rad}} \rangle = \tfrac{3}{4}\hbar\omega(1 - \tfrac{2}{9}\rho^2 + \ldots). \tag{2.16}$$

The radial uncertainty product accordingly approaches

$$\Delta r \, \Delta p_r = \tfrac{1}{2}\hbar(9 - 24/\pi)^{1/2}(1 + \tfrac{2}{9}\rho^2 + \ldots) \tag{2.17}$$

for small radii. The numerical value of the square root, 1.1664, is only slightly larger than the minimum allowed by the uncertainty relation (1.1).

2.4. Large radii

The classical limit of large radii results from the asymptotic expansion of the confluent hypergeometric function (AS13.5.1),

$$\langle r^n \rangle \to R^n\left(1 + \frac{n(n+1)}{4\rho^2} + \frac{(n-2)(n-1)n(n+1)}{32\rho^4} + \ldots\right) - \frac{\delta_{n,-1}}{(2\pi\chi)^{1/2}\rho^2}\exp(-\rho^2)(1 + \ldots). \tag{2.18}$$

Here the second term applies only for $n = -1$ as the lowest-order correction. Hence

$$\langle r \rangle \to R(1 + 1/2\rho^2) \tag{2.19}$$

and

$$(\Delta r)^2 \to \chi(1 - 1/2\rho^2). \tag{2.20}$$

Similarly,

$$\langle p_r^2 \rangle = (\Delta p_r)^2 = (\hbar^2/2\chi)(1 + 1/4\rho^2 + \ldots) \tag{2.21}$$

$$\langle T_{\mathrm{rad}} \rangle = \tfrac{1}{2}\hbar\omega(1 + 1/4\rho^2 + \ldots). \tag{2.22}$$

The radial uncertainty product therefore tends towards

$$\Delta r \, \Delta p_r = \tfrac{1}{2}\hbar\sqrt{2}(1 - 1/8\rho^2 + \ldots) \tag{2.23}$$

on going to large distances. Here again the uncertainty product is only slightly larger than its minimum. Both limits, (2.17) and (2.23), can be seen in figure 1.

3. Azimuthal angle uncertainties

3.1. Angular momenta

Here we employ an unconventional method for the evaluation of expectation values of powers of l_z. Using R and Φ as generators of the wavefunction (1.8) and denoting an arbitrary operator by A, one obtains the identities

$$\hbar(\partial/\partial\Phi)\langle A\rangle = \mathrm{i}\langle l_z A - A l_z\rangle, \tag{3.1}$$

$$\hbar[2\rho^2 + \rho(\partial/\partial\rho)]\langle A\rangle = \langle l_z A + A l_z\rangle. \tag{3.2}$$

Equation (3.2) can be rewritten by using the classical angular momentum (1.7*b*) as generator. This results in the recursion relation

$$\langle l_z^{n+1}\rangle = L[1 + \hbar(\partial/\partial L)]\langle l_z^n\rangle, \qquad n \geqslant 0. \tag{3.3}$$

Starting with $n = 0$, we get

$$\langle l_z\rangle = L, \tag{3.4a}$$

$$\langle l_z^2\rangle = L(L+\hbar), \tag{3.4b}$$

and so on. The uncertainty of the z component of the angular momentum therefore reads

$$\Delta l_z = (\hbar L)^{1/2} = \hbar\rho. \tag{3.5}$$

Expectation values of the other components of the angular momentum vanish by symmetry arguments,

$$\langle l_x\rangle = \langle l_y\rangle = 0, \tag{3.6}$$

and the square of the angular momentum has the expectation value

$$\langle l^2\rangle = L(L+2\hbar). \tag{3.7}$$

Note the difference between (3.7) and the usual result $L(L+\hbar)$ for a harmonic oscillator in an eigenstate. Finally, the expectation value of the centrifugal energy,

$$\langle T_{\mathrm{centr}}\rangle = (1/2m)\langle l^2/r^2\rangle = \tfrac{1}{2}m\omega^2R^2 + \tfrac{1}{4}\hbar\omega(1 - D(\rho)), \tag{3.8}$$

is calculated by explicit integration. Together with the radial kinetic energy (2.12), the total kinetic energy becomes

$$\langle T\rangle = \tfrac{1}{2}m\omega^2R^2 + \tfrac{3}{4}\hbar\omega. \tag{3.9}$$

It contains the other half of the zero-point energy.

3.2. Angular variable

As mentioned in § 1, the operator corresponding to the azimuthal angle variable is defined via its Fourier expansion. Generally, if $f(\phi)$ is a non-periodic function of ϕ, its analogue which is periodic mod 2π is defined by

$$f_{2\pi}(\phi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\phi}\int_0^{2\pi} \mathrm{d}\phi'\, f(\phi')\, \mathrm{e}^{-\mathrm{i}n\phi'}. \tag{3.10}$$

Here we located the discontinuity arbitrarily at $\phi = 0 \bmod 2\pi$. The following quantities are of interest:

$$\phi_{2\pi}(\phi) = \pi - 2 \sum_{n=1}^{\infty} \frac{\sin n\phi}{n}, \tag{3.11a}$$

$$(\phi^2)_{2\pi}(\phi) = 2\pi\phi_{2\pi}(\phi) - \tfrac{2}{3}\pi^2 + 4 \sum_{n=1}^{\infty} \frac{\cos n\phi}{n^2}, \tag{3.11b}$$

$$\delta_{2\pi}(\phi) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos n\phi, \tag{3.11c}$$

where $\delta_{2\pi}$ is the periodic delta function. Note that, at the discontinuity, $\phi_{2\pi}$ and $(\phi^2)_{2\pi}$ take on the definite values

$$\phi_{2\pi}(0) = \pi, \qquad (\phi^2)_{2\pi}(0) = 2\pi^2. \tag{3.12}$$

Combining (3.11a) and (3.11c) we obtain the commutator

$$[\phi_{2\pi}, l_z] = \mathrm{i}\hbar E_{2\pi}, \tag{3.13}$$

$$E_{2\pi}(\phi) = 1 - 2\pi\delta_{2\pi}(\phi), \tag{3.14}$$

so that the uncertainty relation (1.1) becomes

$$\Delta\phi_{2\pi}\Delta l_z \geq \tfrac{1}{2}\hbar|\langle E_{2\pi}\rangle|. \tag{3.15}$$

Evaluation of the integral $\langle E_{2\pi}\rangle$, which is a function of Φ and ρ and replaces unity in our example, proceeds via modified Struve functions (GR.387.5) which, in turn, can be integrated with the help of the complementary error function (GR.6.825) $\mathrm{erfc}(-x) = 1 + \mathrm{erf}(x)$:

$$\langle E_{2\pi}\rangle = 1 - \exp(-\rho^2) - \sqrt{\pi}\rho \cos\Phi \exp(-\rho^2 \sin^2\Phi)\, \mathrm{erfc}(-\rho\cos\Phi). \tag{3.16}$$

Expectation values $\langle\phi_{2\pi}\rangle$ and $\langle(\phi^2)_{2\pi}\rangle$ could, in principle, be obtained by integrating the recursion relation (3.1) with respect to Φ:

$$(\partial/\partial\Phi)\langle\phi_{2\pi}\rangle = \langle E_{2\pi}\rangle, \tag{3.17a}$$

$$(\partial/\partial\Phi)\langle(\phi^2)_{2\pi}\rangle = 2\langle\phi_{2\pi}\rangle + 2\pi\langle E_{2\pi}\rangle - 2\pi. \tag{3.17b}$$

Integrals of this type, however, do not exist in terms of known elementary or special functions. According to their definitions (3.11a, b) we therefore derive

$$\langle\cos n\phi\rangle = C_n(\rho)\cos n\Phi, \qquad \langle\sin n\phi\rangle = C_n(\rho)\sin n\Phi, \tag{3.18}$$

where the functions

$$C_n(\rho) = \frac{\sqrt{\pi}}{2^n\Gamma((n+1)/2)}\rho^n \exp(-\rho^2) M\left(\frac{n+2}{2}, n+1; \rho^2\right)$$
$$= \tfrac{1}{2}\sqrt{\pi}\rho \exp(-\rho^2/2)(I_{(n-1)/2}(\rho^2/2) + I_{(n+1)/2}(\rho^2/2)) \tag{3.19}$$

are given in terms of modified Bessel functions. Hence

$$\langle\phi_{2\pi}\rangle = \pi - 2\sum_{n=1}^{\infty} C_n(\rho)\frac{\sin n\Phi}{n}, \tag{3.20a}$$

$$\langle(\phi^2)_{2\pi}\rangle = 2\pi\langle\phi_{2\pi}\rangle - \tfrac{2}{3}\pi^2 + 4\sum_{n=1}^{\infty} C_n(\rho)\frac{\cos n\Phi}{n^2}. \tag{3.20b}$$

Special cases can be solved explicitly:

$$\langle\phi_{2\pi}\rangle_{\Phi=0}=\langle\phi_{2\pi}\rangle_{\Phi=\pi}=\pi, \tag{3.21a}$$

$$\langle\phi_{2\pi}\rangle_{\Phi=\pi\pm\pi/2}=\pi\pm\tfrac{1}{2}\pi\ \mathrm{erf}(\rho), \tag{3.21b}$$

$$\langle\phi_{2\pi}\rangle_{R=0}=\pi, \qquad \langle(\phi^2)_{2\pi}\rangle_{R=0}=\tfrac{4}{3}\pi^2. \tag{3.21c}$$

Equations (3.20*a*, *b*) were solved numerically, with the result that both functions approach their limiting values $\Phi_{2\pi}(\Phi)$ and $(\Phi^2)_{2\pi}(\Phi)$ respectively, on going to large radii. For medium radii they resemble incomplete Fourier expansions which tend to constants for small radii. The resulting uncertainty product in units of $\hbar/2$ is plotted in figures 2 and 3. Apart from the points of discontinuity, $\Phi=0, 2\pi$ it is a smooth function of angle and radius.

3.3. *Small radii*

The expressions (3.4)–(3.7) are exact for all radii. Expectation values and uncertainties of all components of the angular momentum therefore vanish on going to small radii. Also, the expectation value of the centrifugal energy,

$$\langle T_{\mathrm{centr}}\rangle=\tfrac{2}{3}\hbar\omega\rho^2+\ldots, \tag{3.22}$$

tends to zero, and the zero-point fluctuations of the kinetic energy are all contained in the radial kinetic energy (2.16).

Limiting values for the angle-dependent quantities are derived from the series expansions of the error function (AS7.15) and of the modified Bessel function (AS9.6.10):

$$\langle E_{2\pi}\rangle=-\sqrt{\pi}\rho\cos\Phi+\ldots, \tag{3.22a}$$

$$\langle\phi_{2\pi}\rangle=\pi-\sqrt{\pi}\rho\sin\Phi+\ldots, \tag{3.22b}$$

$$\langle(\phi^2)_{2\pi}\rangle=\tfrac{4}{3}\pi^2+2\sqrt{\pi}\rho(\cos\Phi-\pi\sin\Phi)+\ldots, \tag{3.22c}$$

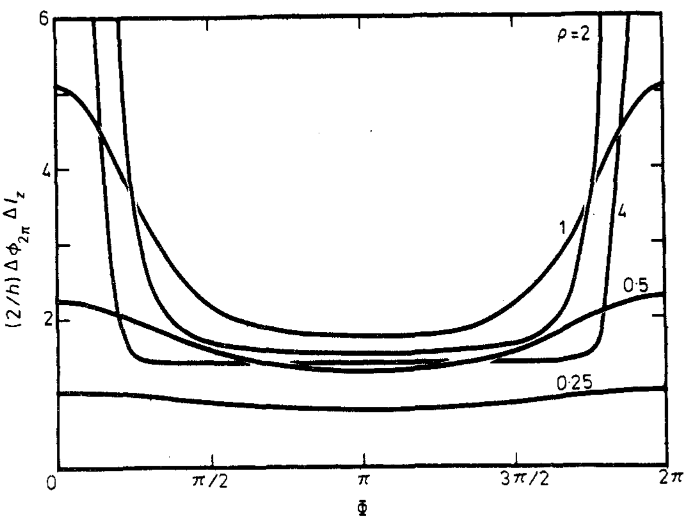

Figure 2. Uncertainty product in units of $\hbar/2$ of the angular variable and z component of the angular momentum, plotted against the classical angle $\Phi=\omega t$ at various classical radii.

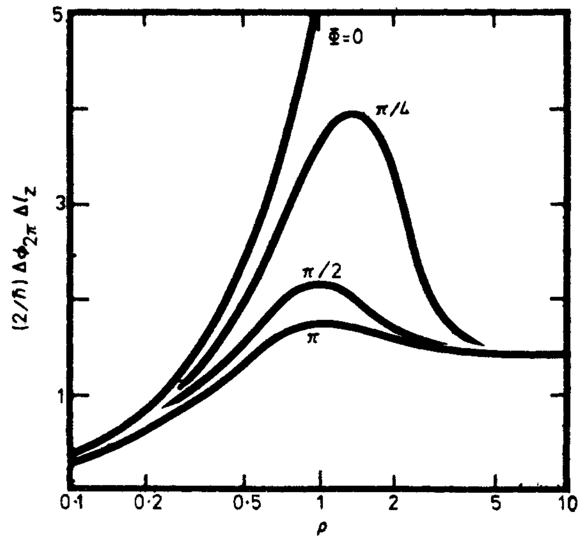

Figure 3. Same as figure 2, but plotted against the classical radius at various classical angles.

According to (3.15) and (3.22), the uncertainty product therefore may vanish for vanishing radius. This is indeed the case, as can be seen from

$$\Delta\phi_{2\pi} = (\pi/\sqrt{3})(1 + 3\pi^{-3/2}\rho\cos\Phi + \dots), \tag{3.23a}$$

$$\Delta\phi_{2\pi}\,\Delta l_z = (\pi/\sqrt{3})\hbar\rho + \dots. \tag{3.23b}$$

However, this applies only to the immediate vicinity of $R = 0$. For radii $R > 0.39\sqrt{\chi}$, i.e. of the order of the ground state fluctuation, the uncertainty product is already larger than $\hbar/2$.

3.4. Large radii

The asymptotic form of the expectation value of the centrifugal energy reads

$$\langle T_{\text{centr}}\rangle = \tfrac{1}{2}m\omega^2 R^2 + \tfrac{1}{4}\hbar\omega(1 - 1/2\rho^2 + \dots), \tag{3.24}$$

and it can be observed that the zero-point fluctuation in the kinetic energy has shifted partially to the centrifugal energy.

The angular momentum uncertainty tends towards infinity, and hence, unless the angular uncertainty tends to zero, the uncertainty product does not stay infinite. It will be shown below that this is the case for all angles except for the points of discontinuity. With the help of

$$\operatorname{erfc}(-\rho\cos\Phi) \to 2\Theta(\cos\Phi), \qquad \rho\exp(-\rho^2\sin^2\Phi) \to \sqrt{\pi}\,\delta(\sin\Phi)$$

for $\rho \to \infty$, where $\Theta(x)$ is the Heaviside step function and $\delta(x)$ the Dirac delta function, we get

$$\langle E_{2\pi}\rangle \to 1 - 2\pi\delta_{2\pi}(\Phi). \tag{3.25}$$

According to equation (1.1), the uncertainty product therefore becomes infinite at the discontinuity and $\geqslant\hbar/2$ elsewhere. In order to derive the limiting values of (3.20), note that the asymptotic series of the Bessel function (AS9.7.1) yields

$$C_n(\rho) = 1 - \frac{n^2}{4\rho^2} + \frac{(n^2-4)n^2}{2!(4\rho^2)^2} + \dots \tag{3.26}$$

The semiconvergent series (3.20) then can be partially summed to give

$$\langle \phi_{2\pi} \rangle \to \Phi_{2\pi}(\Phi) + \dots, \tag{3.27a}$$

$$\langle (\phi^2)_{2\pi} \rangle \to (\Phi^2)_{2\pi}(\Phi) + 1/2\rho^2 + \dots \tag{3.27b}$$

Here the dots denote further terms like $\delta_{2\pi}(\Phi)/\rho^2$ and derivatives with respect to Φ which arise from resuming a semiconvergent series. From (3.27) and

$$(\Phi^2)_{2\pi} = \begin{cases} 2\pi^2 \\ (\Phi_{2\pi})^2 \end{cases} \quad \text{for} \begin{cases} \Phi = 0 \\ \Phi \neq 0 \end{cases}, \tag{3.28}$$

one gets

$$\Delta\phi_{2\pi} \to \begin{cases} \pi \\ 1/\sqrt{2}\rho \end{cases} \quad \text{for} \begin{cases} \Phi = 0 \\ \Phi \neq 0 \end{cases}. \tag{3.29}$$

Finally, the uncertainty product becomes

$$\Delta\phi_{2\pi}\, \Delta l_z \to \begin{cases} \hbar\pi\rho \\ \hbar\sqrt{2}/2 \end{cases} \quad \text{for} \begin{cases} \Phi = 0 \\ \Phi \neq 0 \end{cases}. \tag{3.30}$$

This limit and the approach to it can be seen in figures 2 and 3.

4. Remark on the polar angle variable

From classical arguments, Blochinzew (1972) defines the polar momentum operator

$$p_\theta = (\hbar/\mathrm{i}) \sin^{-1/2}\theta (\partial/\partial\theta) \sin^{1/2}\theta \tag{4.1}$$

and claims that this operator corresponds to the momentum conjugate to the polar angle θ. However, although

$$\langle \theta \rangle = \pi/2, \qquad \langle p_\theta \rangle = 0 \tag{4.2}$$

by symmetry arguments,

$$\langle p_\theta^2 \rangle = \infty \tag{4.3}$$

for all distances by virtue of $\langle \cot^2\theta \rangle = \infty$. The term 'polar angle uncertainty' therefore has no meaning. From this we conclude that the definition (4.1) is meaningless, which is also confirmed by the fact that there exists no classical observable corresponding to p_θ.

5. Summary and discussion

We calculated various expectation values and uncertainties of radial and angular variables and momenta for a three-dimensional isotropic oscillator in the coherent state. All expectation values of radial quantities depend only on the classical radius R and not on the classical angle $\Phi = \omega t$. The radial uncertainty product is of the order of

$\hbar/2$ for all classical radii. This holds also for the limit of vanishing R, where the oscillator turns into its quantum ground state, and for the limit of very large radii, where the oscillator becomes classical. This proves the usefulness of the concept of radial momentum for this example.

In order to deal with periodic functions only, operators corresponding to powers of the azimuthal angle variable were defined via their Fourier expansions in the interval $0\ldots 2\pi$. Expectation values of these depend on R and Φ, but those of angular momentum components depend on R only. Here the discontinuity at $\Phi = 0 \bmod 2\pi$ manifests itself in a large uncertainty product at this point, which even becomes infinitely large for $R \to \infty$. In the limit of vanishing radius, on the other hand, the revolution of the particle round the centre turns into rotation about its own axis. The fact that quantum mechanically this is indistinguishable from a particle at rest, manifests itself in a vanishing angle–angular momentum uncertainty product. However, this is only the case for very small radii. If the radius is of the order of the ground state uncertainty or larger, this uncertainty product is about $\hbar/2$ for all angles except for the discontinuity. As a consequence, for all meaningful values of the classical variables R, Φ, the concept of angular operators is meaningful too.

These studies were motivated by the results of Nieto (1967) and Carruthers and Nieto (1968), who showed that for the same wave packet as employed above, the angle–angular momentum uncertainty product stays finite provided that

$$\langle \phi \rangle - \Phi \ll 1. \tag{5.1}$$

As can be seen from the results of § 3, the condition (5.1) holds only for $R \to \infty$ and $\Phi \neq 0 \bmod 2\pi$, i.e. for the classical limit.

Finally, we investigated the proposed definition of a polar momentum operator. Since the expectation value of the square of this operator turns out to be infinitely large, the corresponding uncertainty is not a useful quantity.

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